Metric extension and embedding theorems

Autumn Semester 2021

The seminar will cover various extension and embedding theorems for metric spaces. Along the way we will encounter many interesting concepts from metric geometry. We will mainly follow the lecture notes [Hei03] and [Hei04] written by Juha Heinonen. Talks can be held in French, German or English.

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Topics

1 Basic concepts and Kuratowski's embedding

[Hei03], 1.3 – 1.19. Introduction of useful tools, such as maximal nets and Lipschitz partition of unity. Proof of the Kuratowski embedding theorem and its application, Proposition 1.9.

2 Fréchet embedding and Urysohn universal space

[Hei03], 1.22 and 3.1 – 3.5. Proof of Fréchet's embedding theorem. Definition of the Urysohn universal space and proof of its existence and uniqueness. It is only 4 pages of material and the idea is that the proofs are given as detailed as possible.

3 Banach's embedding theorem

[Hei03], 3.6 – 3.8 and 3.12. Proof of Banach's embedding theorem by the use of Theorem 3.8, which states that every compact metric space is a continuous image of the Cantor set. If time permits, Theorem 3.12 (Aharoni's embedding theorem) can also be discussed.

4 Gromov's embedding and compactness theorems

[Lan07], 1.2 – 2.4. Proof of Gromov's embedding theorem (Theorem 1.3) and definition of Hausdorff distance. Proof of Blaschke's theorem (Theorem 2.1). Definition of Gromov-Hausdorff distance.

5 The Kirszbraun–Valentine theorem

[Hei04], 2.1 – 2.2. McShane-Whitney extension theorem (Theorem 2.3) and detailed proof of the Kirszbraun–Valentine theorem, which states that every 1-Lipschitz map $f\colon A \to \mathbb{R}^m$, $A \subset \mathbb{R}^n$, can be extended to a map $\bar{f}\colon \mathbb{R}^n \to \mathbb{R}^m$.

6 Lipschitz retracts in $\mathbb{R}^n$

[Hei04], 2.4 – 2.6 without Theorem 2.11. Definition of Lipschitz retract and equivalence to Lipschitz extension (Proposition 2.10). Proof of Hohti's theorem (Theorem 2.12).

7 Hyperconvex metric spaces

[EK01], Definition 2.5, Chapter 3, Subsections 4.1 – 4.2 and Theorem 6.1. Definition of hyperconvex metric spaces and basic examples. Admissible subsets and externally hyperconvex sets. A space is hyperconvex if and only if it is injective (Theorem 4.2). Fixed point results for non-expansive mappings on hyperconvex spaces (Theorem 6.1).

8 Isbell's injective hull of metric spaces

[Lan13], Section 3 up to Proposition 3.5. Definition of $\Delta(X)$ and $E(X)$. Construction of the map $p\colon \Delta(X) \to E(X)$. Proof that $E(X)$ is the injective hull of X (Theorem 3.3). Various characterizations of the injective hull (Proposition 3.4).

9 Rademacher's theorem

[Hei04], pp. 18–23. Detailed proof of Rademacher's theorem, which states that every Lipschitz function $f\colon U \to \mathbb{R}^m$, $U \subset \mathbb{R}^n$ open, is differentiable almost everywhere.

10 Bourgain's embedding theorem

[Bou85], Propositions 1 and 3. Proof of Bourgain's embedding theorem, which states that every n-point metric space is $C \cdot \log n$ lipeomorphic to a subset of $\mathbb{R}^n$.

11 Assouad's embedding theorem

[Hei03], 3.14 – 3.16. Definition of doubling metric spaces (see 1.14 in Section 1 of the lecture notes). Examples 3.16(b)–(d). Proof of Assouad's embedding theorem.

12 Lipschitz neighborhood retracts

[AJ62], Section 1 up to Proposition 1.4. Proof of a result due to Federer, which gives a sufficient criterion for a subset $A \subset \mathbb{R}^n$ to be a Lipschitz neighborhood retract.

References

[AJ62] F. J. Almgren Jr. The homotopy groups of the integral cycle groups. *Topology*, 1(4):257–299, 1962.

[Bou85] J. Bourgain. On lipschitz embedding of finite metric spaces in hilbert space. *Israel Journal of Mathematics*, 52(1-2):46–52, 1985.

[EK01] R Espínola and Mohamed A Khamsi. Introduction to hyperconvex spaces. In *Handbook of metric fixed point theory*, pages 391–435. Springer, 2001. URL: `http://drkhamsi.com/publication/Es-Kh.pdf`.

[Hei03] J. Heinonen. Geometric embeddings of metric spaces. *Lecture Notes*, 2003. URL: `https://jyx.jyu.fi/bitstream/handle/123456789/22520/rep90.pdf?sequence=1&isAllowed=y`.

[Hei04] J. Heinonen. Lectures on lipschitz analysis. *Lecture Notes*, 2004. URL: `http://www.math.jyu.fi/research/reports/rep100.pdf`.

[Lan07] U. Lang. Notes on rectifiability. *Lecture Notes*, 2007. URL: `https://people.math.ethz.ch/~lang/rect_notes.pdf`.

[Lan13] U. Lang. Injective hulls of certain discrete metric spaces and groups. *Journal of Topology and Analysis*, 5(03):297–331, 2013. URL: `https://arxiv.org/pdf/1107.5971.pdf`.